Kamnitzer

$Gr := G(K)/G(\mathcal{O})$ $\qquad \pi_0(Gr) = \pi_1(G) = \Lambda/\mathbb{Z}R$

$Gr^\lambda := G(\mathcal{O})\cdot t^\lambda \qquad \lambda \in \Lambda_+$

$\overline{Gr^\lambda} = \bigcup_{\mu \leqq \lambda} Gr^\mu$

Eg. $G = SL_2$ $\qquad \mathbb{Z}(1,-1)$

$\Lambda = \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = 2\mathbb{Z} \qquad R = \{2, -2\}$

$\Lambda_+ = 2\mathbb{Z}_+ = \{0, 2, 4, \dots\} \qquad \leqq$ usual order

$Gr^0 \subset \overline{Gr^2} \subset \overline{Gr^4}$

dim $\quad 0 \quad 2 \quad 4$

$G = PGL_2$

$\Lambda = \mathbb{Z}^2/\mathbb{Z}(1,1) \qquad R = \{(1,-1), (-1,1)\}$

$= \mathbb{Z} \qquad = \{2, -2\}$

$\Lambda_+ = \mathbb{Z}_+ \qquad \mu \leqq \lambda \Leftrightarrow \lambda - \mu = \sum n_i \alpha_i^\vee$

$\therefore \quad 0 \leqq 2 \leqq 4 \leqq \cdots$

$\quad 1 \leqq 3 \leqq 5 \leqq$

$\pi_0(Gr) = \mathbb{Z}/2$

$Gr^0 \subset \overline{Gr^2} \subset \overline{Gr^4} \subset \cdots$

$0 \quad 2 \quad 4$

$Gr^1 \subset \overline{Gr^3} \subset \overline{Gr^5} \subset$

$\overline{Gr^1} = \mathbb{P}^1$ (with $Gr^1 = \overline{Gr^1}$)

$G = GL_2 \qquad \Lambda = \mathbb{Z}^2 \qquad R = \{(1,-1), (-1,1)\}$

$\Lambda_+ = \{(\lambda_1, \lambda_2) \mid \lambda_1 \geqq \lambda_2\} \qquad \pi_0(Gr) \cong \mathbb{Z}$

$Gr^0 \subset \overline{Gr}^{(1,-1)} \subset \overline{Gr}^{(2,-2)} \subset \cdots$

$Gr^{(1,0)} \subset \overline{Gr}^{(2,-1)} \subset \cdots$

$\vdots$

$P_{G(\mathcal{O})}(Gr)$ semisimple category

$\left(\text{cf. } P_B(G/B) \cong \mathcal{O}_0(\mathfrak{g}) \right.$ Beilinson – Bernstein localization

simple objects of $P_{G(\mathcal{O})}Gr$ are $IC_{\overline{Gr^\lambda}} =: IC_\lambda$

Gr^λ : simply connected !

$$\{[g,L] \mid g \in G(K), L \in Gr\}$$

$$Gr \tilde{\times} Gr = G(K) \times_{G(\mathcal{O})} Gr \xrightarrow{m} Gr$$

$$[g,L] \mapsto [g] \qquad \downarrow p \qquad Gr$$

$$[g.L] \mapsto g \cdot L$$

(m,p)

$(\cong Gr \times Gr$

This is a Gr-bdle map over Gr via p has the str. group $G(\mathcal{O})$

So given

$A, B \in P_{G(\mathcal{O})}Gr$ form $A \tilde{\boxtimes} B \in P(Gr \tilde{\times} Gr)$

Then define $A * B = m_*(A \tilde{\boxtimes} B)$

($\because$ m is stratified semisimple)

Eg. $A = IC_\lambda$, $B = IC_\mu$

$$A \tilde{\boxtimes} B = IC_{\overline{Gr^\lambda \tilde{\times} Gr^\mu}}$$

$$Gr^\lambda \tilde{\times} Gr^\mu \subset Gr \tilde{\times} Gr$$

$$= \{[g,L] : [g] \in Gr^\lambda \text{ \& } L \in Gr^\mu\}$$

Then $IC_\lambda * IC_\mu = m_*(IC_{\overline{Gr^\lambda \tilde{\times} Gr^\mu}})$

$$= \bigoplus_\nu IC_{\overline{Gr^\nu}} \otimes M^\nu_{\lambda\mu}$$

multiplicity space

Take stalks at t^ν

$$H_{top}(m^{-1}_{\lambda\mu}(t^\nu)) = M^\nu_{\lambda\mu}$$

$$m_{\lambda\mu} : \overline{Gr^\lambda \tilde{\times} Gr^\mu} \longrightarrow Gr$$

is the restriction of

$$m : Gr \tilde{\times} Gr \longrightarrow Gr$$

(Lusztig, Ginsburg, Mirkovic-Viloven)

Th. There exists an equivalence of tensor categories compatible with fiber functors

$$\begin{tikzcd} A \in \mathcal{P}_{G(\mathcal{O})} Gr \arrow[rr, "\sim"] \arrow[dr] & & \mathrm{Rep}\, G^\vee \arrow[dl, "\text{forget}"] \\ & \mathrm{Vect} & \end{tikzcd}$$

$A \longmapsto H^*(Gr, A) \in \mathrm{Vect}$

exact

$$IC_\lambda \longleftrightarrow V_\lambda \qquad \lambda \in \Lambda_+$$

$$\downarrow$$

$$H^*(Gr, IC_\lambda) = IH(\overline{Gr^\lambda}) \cong V_\lambda$$

Ex. ① $G = GL_n \quad \lambda = (\overbrace{1 \cdots 1}^{k}, \overbrace{0 \cdots 0}^{n-k}) = \omega_k$

$\overline{Gr^\lambda} - Gr^\lambda$ bottom (periodic)

$= Gr(k, n)$

$IH(Gr(k,n)) \cong V_{\omega_k} = \Lambda^k \mathbb{C}^n$

$\|$

$H(Gr(k,n))$ basis given by Schubert variety indexed by k elements in $\{1, \cdots, n\}$

$\Lambda^k \mathbb{C}^n$ also has an (obvious) basis.

② Compare tensor product

$$IC_\lambda \longleftrightarrow V_\lambda$$

$$IC_\lambda * IC_\mu \longleftrightarrow V_\lambda \otimes V_\mu$$

$$\shortparallel$$

$$m_*(IC_\lambda \tilde{\boxtimes} IC_\mu) = m_*(IC_{\overline{Gr^\lambda \tilde{\times} Gr^\mu}})$$

$$\therefore \; IH(\overline{Gr^\lambda \tilde{\times} Gr^\mu}) \cong V_\lambda \otimes V_\mu$$

$$IH(\overline{Gr^{\lambda_1} \tilde{\times} \cdots \tilde{\times} Gr^{\lambda_n}}) = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

$$Gr \tilde{\times} \cdots \tilde{\times} Gr = G(K) \times_{G(\mathcal{O})} G(K) \times \cdots \times_{G(\mathcal{O})} Gr$$

If lucky $\quad \overline{Gr^\lambda}$ smooth $\quad (\overline{Gr^\lambda} = Gr^\lambda)$

$$\Leftrightarrow V_\lambda \text{ is minuscule}$$

$$H_*(Gr^{\lambda_1} \tilde{\times} \cdots \tilde{\times} Gr^{\lambda_n}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

(all weight in V_λ are Weyl grp orbit of dominant h. wt.

Y_n (in the last week)

$\shortparallel$ $Gr^1 \tilde{\times} \cdots \tilde{\times} Gr^1 \qquad Gr^1 \subset Gr_{GL_2}$

use $Gr_{GL_n} = \{\mathcal{O}\text{-lattice in } K^n\}$

③ Compare tensor product multiplicity

$$IC_\lambda * IC_\mu \longleftrightarrow V_\lambda \otimes V_\mu$$

$$\bigoplus_\nu IC_\nu \otimes \underbrace{M_{\lambda\mu}^\nu}_{= H_{top}(m_{\lambda\mu}^{-1}(t^\nu))} \qquad \bigoplus_\nu V_\nu \otimes \mathrm{Hom}(V_\nu, V_\lambda \otimes V_\mu)$$

$$H_{top}(m_{\lambda\mu}^{-1}(t^{\nu})) = \mathrm{Hom}(V_\nu, V_\lambda \otimes V_\mu)$$

ξ

Spanned by
top dim.
irr. components of $m_{\lambda\mu}^{-1}(t^{\nu})$

$top = \langle \lambda+\mu-\nu, \rho \rangle$

$$H_{top}(F_{(k,k)}) \cong (\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{k})^{SL_2}$$

Springer fiber
to the (k,k) nilpotent

<u>Question</u>

$$\mathcal{P}_{G(\mathcal{O})} Gr \cong \mathrm{Rep}\, G^\vee \longrightarrow \mathrm{Rep}\, T^\vee$$

To see the restriction functor $P_{G(\mathcal{O})}(Gr) \to \mathrm{Rep}\, T^\vee$

We want $T^\vee$ to act $H^*(Gr, A)$ $\quad A \in P_{G(\mathcal{O})} Gr$

" $\quad IH(\overline{Gr}^\lambda)$

We want a grading of $IH(\overline{Gr}^\lambda)$ by Λ

$$V_\lambda = \bigoplus_\mu V_\lambda(\mu)$$

B : Borel of G

N : nilpotent radical

$N(K) \curvearrowright Gr \qquad S^\mu = N(K) \cdot t^\mu \qquad$ semi-infinite

$\qquad Gr = \bigcup S^\mu$

<u>Th</u> ([MV])

For all $A \in \mathcal{P}_{G(\mathcal{O})} Gr$, $H_c^k(S^\mu, A) = 0$ if $k \neq 2\langle \mu, \rho \rangle$

and $\quad H(Gr, A) \cong \bigoplus_\mu H_c^{2\langle \mu,\rho \rangle}(S^\mu, A)$

$$\Rightarrow IH(\overline{Gr^\lambda}) \cong \bigoplus_\mu H_c^{2\langle\mu,\rho\rangle}(S^\mu, IC_\lambda)$$

$$= \bigoplus_\mu H_{top}(\overline{Gr^\lambda} \cap S^\mu)$$

$$= \bigoplus_\mu \mathbb{C}\{\text{components of } \overline{Gr^\lambda} \cap S^\mu\}$$

$\{$ MV cycles

$\overline{Gr^\lambda} = G(k, n)$ atzia Schubert cells